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On self-similar solutions in disc accretion problems

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The theory of disc accretion has recently become an important factor for the solution of many astrophysical problems. Accretion discs are considered to play a major role in the modelling of quasars, nuclear active galaxies and X-ray sources in narrow binary stars. We have enough grounds to claim that the structure of stationary thin accretion gas discs is relatively well studied [15].

Regardless of the progress in these studies, the fundamental problem of the disc accretion theory, i. e. of its viscosity nature, is still to be resolved. Obviously, theoretical investigations are not sufficient. Regular observation data should be used on a broader scale accompanied by comparisons between theoretical models and experimental results.

The non-stationary disc accretion is defined in many cases, where discs are assumed to exist or are observed in the studied objects. Therefore, the investigation of this type of solutions will present the basis of understanding the natural physical phenomena and processes. The study of non-stationary discs provides possibility to make scientifically justified conclusions about the nature of the viscous mechanisms responsible for the transport of the motion quantity momentum in the discs.

Many publications on non-stationary disc accretion are devoted to these problems [8, 9, 10, 11, 12, 13].

In another publication of ours we have found out that the temporal behaviour of thin accretion discs may be described with the non-linear differential maintainance of "diffusion":

$$\frac{\partial F}{\partial t}=A\frac{F^m}{h^n}\frac{\partial^2 F}{\partial h^2}, \tag{1}$$

where h is the specific angular momentum and F is the friction momentum between two adjacent cylindrical layers in the disc [12]; the parameters m and n are determined by the viscosity nature and the opacity law of the disc [10]. A is a "diffusion" coefficient determining the velocity of the processes.

A non-linear diffusion equation of a simpler type has particular invariant self-similar solutions. It is known that the idea of self-similarity is related to the transformation groups [1, 3, 4, 5, 7]. These transformations are represented into the differential or the integrodifferential equations of the process. The group of transformations for the given equation is determined by the dimensions of its input values: time reference unit, length, mass, etc., which represent a simple case.

This type of self-similar solutions is featured with power indices, which represent simple quotients, determined elementary by dimensional analysis operations and known as first order solutions.

We shall define further the role of these self-similar solutions and shall demonstrate the pattern of their application into viscosity problem solutions. Self-similarity is a phenomenon whose features are obtained at different time moments by the transformation of similarities in sequential order. The scales of similarity, in turn, represent a function of the main physical parameters of the equation describing the physical phenomenon.

Let us examine the temperature diffusion equation for stationary conductive medium:

$$\frac{\partial T}{\partial t} = Q \nabla T,$$

where Q is the constant diffusion, T is the temperature and t is the time. The problem is to determine the temperature in the successive moments, if the initial distribution is $T = Br^{\beta}$, where r is the distance to the centre of the coordinate system. If we define the scale of the temperature θ, the distance L and the time τ then we can determine dimensions Q and B: $[Q] = \tau^{-1}L^2$ and $[B] = L^{-\beta}\theta$.

Q is the only constant independent of θ. The problem is precisely determined and there is no other constant of length or time dimension to be obtained from the elements given above. Therefore, such a constant should not be present in the solution. Sometimes, after the beginning of the process, the typical length scale depending on the time may be defined as:

$$L_c(t) = (Qt)^{1/2}.$$

The time-dependent temperature scale may be defined in a similar way:

$$T_c(t) = BL_c(t)^{\beta}.$$

The solution of the problem should yield T as a function of t and r. In non-dimensional form this is:

$$\frac{T}{T_c} = \frac{T}{BL^{\beta}}.$$

The non-dimensional form should be a function of $\frac{r}{L_c(t)}$ and $\frac{t}{t}$. The latter is naturally a zero and does not enter the examined problem since t is measured in $[\tau]$ alone and may be expressed by Q, B and t. Thus, we obtain the solution in the form of $T = BL_c^{\beta} T_* \left(\frac{r}{L_c(t)}\right)$, where $L_c(t)$ is already defined and T_* is a non-dimensional function composed of its non-dimensional arguments. The obtained result is a self-similar solution, since time-dependent scales are used. The temperature scale is always a function of the scale featuring length. It is the self-similarity of the problem which denotes that variable scales of L_c and T_c may be selected, which provides for the possibility to represent the scale of the phenomenon characteristics by a single variable function.

Therefore, the presence of several dimensions of the independent constants, including the boundary conditions of the problem, defines the necessity of a self-similar solution.

Let us examine now several problems where the self-similar solutions are of first order [1,3]. The first problem — the time behaviour of a thin disc — is determined by equation (1) under the assumption that for the initial momentum $t=0$ the distribution is:

$$F = Bh^{\alpha}. \tag{2}$$

The dimensions of all values in equation (1) and the initial condition (2) are:

$$\begin{aligned} &[h] = L^{2}\tau^{-1}; \quad [t] = \tau; \quad [F] = ML^{3}\tau^{-2}; \\ &[A] = M^{-m}L^{-2(n-m+2)}\tau^{2n-m-3}; \\ &[B] = ML^{2(1-\alpha)}\tau^{\alpha-2}, \end{aligned} \tag{3}$$

where τ is the time dimension, M is the mass dimension and L is the length dimension.

Let us determine the typical scale of the total angular momentum $h_c(t)$ and the typical friction momentum scale $F_c(t)$ for each moment $t>0$. The first value is yielded by the dimensional analysis of equation (1), namely:

$$h_c(t) = (AF_c(t)^m t)^{\frac{1}{n+2}}, \tag{4}$$

and for $F_c(t)$ we use the initial distribution:

$$F_c(t) = Bh_c(t)^{\alpha}. \tag{5}$$

Substituting expression (5) into (4), we obtain for h_c:

$$h_c(t) = (AB^m t)^{\frac{1}{n+2-\beta m}}. \tag{6}$$

The solution of the problem yields F as a function of h and t and may be written down in a non-dimensional form as:

$$\frac{F}{F_c} = \frac{F}{Bh_c(t)^{\alpha}} = F_*\left(\frac{h}{h_c}, \frac{t}{t}\right) = F_*\left(\frac{h}{h_c}\right).$$

Therefore, the function F will take the shape of:

$$F(h, t) = Bh_c^{\alpha}(t)F_*\left(\frac{h}{h_c}\right). \tag{7}$$

If we substitute (6) into (7), we shall obtain the dependence in developed form, and using equations (7) and (1) we may write the following equation for F_*:

$$\left(\frac{\alpha}{n+2-\alpha m}\right)F_* - \left(\frac{1}{n+2-\alpha m}\right)\xi\frac{dF}{d\xi} = \varphi(t)\frac{F_*^m}{\xi^n}\frac{d^2F_*}{d\xi^2},$$

where $\xi = \frac{h}{h_c(t)}$. The function $\varphi(t)$ is an expression containing only the time. This function should equal a unit in order to provide a self-similar solution for the above equation. And, indeed, for each distribution of the type (2) this condition is satisfied. Therefore, we may write down equation (8) in the final form of:

$$\left(\frac{\alpha}{n+2-\alpha m}\right)F_* - \left(\frac{1}{n-2-\alpha m}\right)\xi\frac{dF_*}{d\xi} = \frac{F_*}{\xi^m}\frac{d^2F_*}{d\xi^2}. \tag{8}$$

Equation (8) provides the possibility for both qualitative and quantitative description of some non-stationary phenomena in the accretion disc. This feature of equation (8) has been the subject of our other works [10].

The second problem leading to a self-similar solution of first order for equation (1) is the following: the evolution of the accretion disc is described by equation (1) under the condition that the time development of the initial configuration is satisfied throughout the process by the integral of the total substance angular momentum [2], namely:

$$K = 2\pi\int_{r'}^{r''}\Sigma\, hrdr, \tag{9}$$

where Σ is the surface disc density [15], h is the angular momentum and r is the distance to the disc center. Substituting Σ with F similar to the procedure in equation (1) [10], we obtain the following condition:

$$K = 2\pi\int_{h'}^{h''}F^{1-m}h^{1+n}\,dh = \text{const}. \tag{10}$$

Another approach to find the necessary conditions for the availability of self-similar solution of equation (7) is to determine the power indices α and β in solutions of the type:

$$F = Ct^{-\alpha}F_*(\xi), \tag{11}$$

where $\xi = \frac{h}{Bt^{\beta}}$,

which reduces the problem to a routine differential equation of finite conditions.

After substituting equation (11) into (1) and (10), we obtain the following necessary conditions for the availability of a self-similar solution:

$$\alpha = 1, \quad \beta = \frac{1-m}{2+n}; \tag{12a}$$

$$C = \frac{K}{2\pi A} \quad B = \left[\frac{A^{1-m}K^m}{(2\pi)^m}\right]^{\frac{1}{n+2}}. \tag{12b}$$

The equation which defines the function F_* is:

$$F_*^m\frac{d^2F_*}{d\xi} + \left(\frac{1-m}{2+n}\right)\xi^{n+1}\frac{dF_*}{d\xi} + \xi^n F_* = 0. \tag{13}$$

We shall examine below some possible astrophysical phenomena, where this solution can be applied.

If a stationary disc has existed in a binary system prior to a certain moment, and due to some reason the inflow of the normal component ceases, this solution will describe the evolution of the reminder of the disc substance.

The same approach on the applicability of equation (13) can be made in the following manner: if we have a distribution of the type $F=\Phi(h)$ at the initial moment, then it will contribute indirectly to integral (10). The function $F=\Phi(h)$ should simply be determined with respect to h.

We have to underline here that this method provides a possibility to expand the number of the astrophysical problems, which may be resolved by the first problem set up in this paper, since the substantial fact is that the distribution should represent a power law of the specific angular momentum. The unique condition to be met by the solution of equation (13) is to satisfy the law of preserving the quantity of the total angular momentum throughout the time development of the initial configuration. Of course, the best proof of this affirmation will be to compare the solutions of equations (13) and (1) for particular derivatives.

Another type of this problem relates to the modelling of the substance behaviour in accretion discs, when the mass integral is satisfied during the evolution process.

Let us examine again equation (1). Applying by analogy the method used by Sedov [7] for the obtaining of the mass integral with reference to rotating fluid, we can obtain the corresponding algebraic integral for the solution of our problem.

Following the same pattern, let us examine the demensions of the input values:

$$[F]=ML^2\tau^2; \quad [h]=L^2\tau^{-1}; \quad [\Sigma]=ML^{-2}; \tag{14}$$
$$[M]=M; \quad [B]=L^2\tau^{1-\delta}.$$

We are looking for a solution of the type $F=Bt^{-a}F_*(\xi)$, where $\xi=\frac{h}{bt^{\delta}}$.

Following Sedov's approach [7], we introduce a supplementary parameter $[a]=ML^k\tau^s$, where a, k, s are unknown, if no particular consideration on the nature of the phenomenon is involved.

Using in a summarized manner the above mentioned development, we can determine some supplementary values, namely:

(15) a) $v_r=\frac{r}{t}V(\xi)$ radial velocity;

(16) b) $\Sigma=\frac{a}{r^{k+3}t^s}R(\xi)$ surface density;

(17) c) $M=\frac{a}{r^k t^s}M(\xi)$ the mass between two fixed radii.

Performing almost the same computations as in [7], we obtain the mass integral in a final form as:

$$\{(s+2\delta k)M(\xi)-2\pi R(\xi)(V(\xi)-2\delta)\}=\text{const}\,\xi^{2k}. \tag{18}$$

Knowing functions $R(\xi)$ and (ξ), we may determine the temporal mass behaviour between two radii. Using the theory of the disc accretion, these relations take the shape of (14):

$$R(\xi)=\xi^{n+2k+1}F_*(\xi)^{(1-m)}; \tag{19}$$

$$V(\xi)=-\xi^{-(n+1)}F_*^{-(1-m)}\frac{df}{d\xi}. \tag{20}$$

The equation satisfying function $f(\xi)$ is similar to equation (13) but the coefficients differ:

$$F_*^m \frac{d^2F_*}{d\xi^2} + \delta\xi^{n+1}\frac{dF_*}{d\xi} + \alpha\xi^n F_* = 0, \qquad (21)$$

where α, δ are defined by dimensional analysis.

Let us examine the values v_r and Σ in view of the disc accretion theory, using the dependences which are relevant to each and any moment. Our purpose is to determine the dependence between the dimension coefficients and the power indices. As a final result we obtain the following equation system defining the power indices:

$$\begin{aligned} &2k\delta + s + 1 = 0; \\ &\delta(2k+n+1) - \alpha(1-m) + s = 0; \\ &2k\delta + \delta + 1 - \alpha - s = 0; \\ &2\delta + \delta n + \alpha m - 1 = 0. \end{aligned} \qquad (22)$$

The solutions corresponding to the system (22) are:

$$\begin{aligned} &\alpha = -\frac{1}{n+2-m}; \quad \delta = \frac{1}{n+2-m}; \\ &K = \frac{1}{2}(m-n-3); \quad s = \frac{1}{n+2-m}. \end{aligned} \qquad (23)$$

Thus, with the power indices obtained and for the finite conditions of the function F_*, the equation (21) yields a solution which describes the temporal behaviour of the accretion disc under the condition that the mass integral is satisfied. The solution of the equation (21) with the power indices (23) may be used for describing the temporal evolution of the substance tore formed around a gravity centre.

In the case where the right-side constant of equation (18) differs from a zero, we shall observe in dependence on the sign either an increase or a decrease of the total tore or disc mass.

We should note that usually in a real time situation both algebraic integrals must be used, i. e. the integral reflecting the situation prior to the moment of quantity motion and the mass integral. When we examine the case of a disc evolution involving contribution of a substance flux inflowing from a secondary component, both the disc mass and the moment of quantity motion change. This imposes the necessity of investigating more complicated problems which will be the subject of further studies.

Discussion

The three methods proposed for the solution of the problems related to equation (1) provide for the possibility of building up both quantitative and qualitative models of non-stationary sources with the assumed existence of accretion discs. The obtaining of a large class of particular solutions and their comparison with the observational data for transient and cataclismic stars provides for the closer understanding of the physical processes of the disc accretion. Examining the methods given here and based on a general analysis approach, as well as comparing them with similar physical methods [1, 4, 5, 7], we can obtain the power index solutions depending on the parameters m and n. In turn, they will provide the definition of the physical processes of the thin accretion disc viscosity and of the plasma opacity.

On the basis of observations of cataclismic stars and transient X-ray sources it is determined that their luminosity after attaining a maximum decreases almost after a power law in time. This provides grounds to believe that by comparing the model solutions and the existing physical hypotheses we can obtain estimation of the scale and nature of the physical processes in the disc.

On the other hand, this paper examines only methods providing for the obtaining of self-similar solutions of first order. Second order soutions are also available [1, 4]. The second order solutions provide for the possibility of estimating new models, thus yielding solutions of this order bised on astrophysical considerations.

In conclusion we shall underline another essential fact. Equation (1) is closely approximating the equations describing the combustion processes and some of the plasma processes [6]. Certain non-routine processes, typical for non-linear plasma and combustion properties are also to be observed within these physical phenomena, i. e. self-organization, self-focusing, etc. This provides grounds to believe that such phenomena may also be expected in disc accretion processes. These aspects deserve specific attention and should become the subject of future investigation of the nature and the properties of the accretion discs.

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Об автомодельных решениях задачи дисковой акреции

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(Резюме)

Гидродинамические уравнения используются как один из методов моделирования нестационарных дисков вокруг одной из компактных компонентов в двойных системах.

Полученные модельные уравнения — нелинейные и могут быть решены при помощи численных или теоретико-групповых методов.

В настоящей работе автор использует модельное уравнение, исходя из предположения, что законы вязкости и непрозрачности являются степенными функциями локальных параметров акреционного диска.

Данные наблюдения нестационарных компактных объектов, где можно ожидать существования акреционных дисков, сравнивали с решениями уравнений, содержащих большинство общих предположений о физических процессах в дисках. Автор считает, что это найдет применение в определении природы и масштаба явлений. Таким образом сформулированны три реальные астрофизические проблемы, считая, что они ведут к автомодельным решениям первого рода, используя уравнения, полученные в настоящей работе.